MATHEMATICS

SLOWING DOWN OF NEUTRONS IN A CYLINDRICAL PILE

BY

V. L. DESHPANDE AND V. M. BHISE

(Communicated by Prof. C. J. BOUWKAMP at the meeting of March 27, 1971)

1. SNEDDON [1, pp. 206–226] has discussed the slowing down of neutrons in matter, with the help of the neutron transport equation which under certain assumptions reduces to the basic equation of age theory. He has discussed the solutions for an infinite as well as a finite pile in the form of a rectangular parallelopiped for various source functions. In this paper, we consider a pile in the form of a cylinder of finite height and obtain the solution for various types of source.

We require the following mathematical results in our discussion.

From [1, p. 74, and p. 83], if $f(x)$ satisfies Dirichlet's conditions in the interval $(0, c)$ and if for that range its finite sine transform is defined to be

$$\bar{f}(q) = \int_0^c f(z) \sin\left(\frac{q\pi z}{c}\right) dz, \tag{1.1}$$

then, at each point of $(0, c)$ at which $f(z)$ is continuous,

$$f(z) = \frac{2}{c} \sum_{q=1}^{\infty} \bar{f}(q) \sin\left(\frac{q\pi z}{c}\right), \tag{1.2}$$

and, if a similar type of function $f(x)$ has a finite Hankel transform defined by

$$f_J(\xi_i) = \int_0^a r f(r)\, J_0(\xi_i r)\, dr, \tag{1.3}$$

where ξ_i is a root of the equation

$$J_0(\xi a) = 0, \tag{1.4}$$

then

$$f(x) = \frac{2}{a^2} \sum_i f_J(\xi_i) \frac{J_0(r\xi_i)}{[J_1(a\xi_i)]^2}, \tag{1.5}$$

where the sum is taken over all the positive roots of equation (1.4).

Also, taking the finite sine transform and finite Hankel transform of

$$\left\{ \begin{aligned} s(r, z) &= r^{\varrho-2}(a-r)^{\nu-1} \sin^{1-2\alpha'}\left(\frac{\pi z}{c}\right) \cdot \\ &\quad \cdot F_4\left[\alpha, \beta; \nu, \delta; (a-r)\sin^2\left(\frac{\pi z}{c}\right), r\sin^2\left(\frac{\pi z}{c}\right)\right], \end{aligned} \right. \tag{1.6}$$

we have

$$\bar{s}_J(\xi_i, q) = \int_0^a rJ_0(\xi_i r) \int_0^c s(r, z) \sin\left(\frac{q\pi z}{c}\right) dz\, dr.$$

Replacing z by $\frac{cz}{\pi}$, q by $2\lambda+1$, expressing the Appell function F_4 in $s(r, z)$ in series form, and interchanging the order of integration and summation, we obtain

$$\bar{s}_J(\xi_i, 2\lambda+1) = \frac{B(\varrho, \nu)\, \Gamma(\frac{3}{2}-\alpha')\, \Gamma(\alpha'+\lambda)}{\sqrt{\pi}\Gamma(\alpha')\Gamma(2-\alpha'+\lambda)} \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}\, (\beta)_{m+n}}{m!n!(\varrho+\nu)_{m+n}}$$

$$\frac{(\frac{3}{2}-\alpha')_{m+n}(1-\alpha')_{m+n}(\varrho)_n(a)^{\varrho+\nu+m+n}}{(1-\alpha'-\lambda)_{m+n}(2-\alpha'+\lambda)_{m+n}\,(\delta)_n} \cdot {}_2F_3\left[\begin{matrix}\Delta(2, \varrho+n), \\ 1, \Delta(2, \varrho+\nu+m+n),\end{matrix} \; -\frac{\xi_i^2 a^2}{4}\right]$$

with the help of [2, p. 193, (56)] and [3, p. 15 (4)], λ being a positive integer.

Further, expressing ${}_2F_3$ in series form, and using [3, p. 9 (10)], we obtain, on simplification, the formula required:

$$\bar{s}_J(\xi_i, 2\lambda+1) = \frac{B(\varrho, \nu)\, \Gamma(\frac{3}{2}-\alpha')\, \Gamma(\alpha'+\lambda)\, a^{\varrho+\nu}}{\Gamma(\alpha')\, \Gamma(2-\alpha'+\lambda)\sqrt{\pi}} \cdot$$

$$\cdot \sum_{s=0}^{\infty} \frac{(\varrho)_{2s} \left(-\frac{\xi_i^2 a^2}{4}\right)^s}{(s!)^2(\varrho+\nu)_{2s}} \; F^{4,1}_{3,1}\left[\begin{matrix} a \\ a \end{matrix}\left|\begin{matrix}\alpha, \beta, \frac{3}{2}-\alpha', 1-\alpha' : \text{———} ; \varrho+2s \\ \varrho+\nu+2s, 1-\alpha'-\lambda, 2-\alpha'-\lambda : - ; \delta\end{matrix}\right.\right],$$

provided $\mathrm{Re}\,(\nu)>0$, $\mathrm{Re}\,(\varrho)>0$, $\mathrm{Re}\,(\alpha')<1$.

2. The equation governing the slowing down density $\chi(r, \phi, z, \theta)$ of neutrons in cylindrical polar coordinates is [1, p. 221, (63)]

$$\frac{\partial\chi}{\partial\theta} = \frac{\partial^2\chi}{\partial r^2} + \frac{1}{r}\frac{\partial\chi}{\partial r} + \frac{\partial^2\chi}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2\chi}{\partial\phi^2} + T(r, \phi, z, \theta), \tag{2.1}$$

where $T(r, \phi, z, \theta)$ is the known source of neutrons in the material and θ is the symbolic age [1, p. 214, (35)]. Assuming symmetry about the axis of the cylinder, we have

$$\frac{\partial\chi}{\partial\theta} = \frac{\partial^2\chi}{\partial r^2} + \frac{1}{r}\frac{\partial\chi}{\partial r} + \frac{\partial^2\chi}{\partial z^2} + T(r, z, \theta), \tag{2.2}$$

where $\chi \equiv \chi(r, z, \theta)$.

Consider the case of a finite cylinder of radius a and bounded by the planes $z=0$ and $z=c$. We assume boundary conditions such that the slowing down density vanishes on the boundary of the cylinder, i.e., $\chi(a, z, \theta) = \chi(r, 0, \theta) = \chi(r, c, \theta) = 0$, and suppose the source function to be $T(r, z, \theta) = s(r, z)\delta(\theta)$, where $\delta(\theta)$ is the Dirac delta function.

Multiplying (2.2) throughout by $\sin\left(\frac{q\pi z}{c}\right)$, integrating with respect to

z from $z=0$ to $z=c$ and using the boundary conditions, we have

$$\frac{\partial \bar{\chi}}{\partial \theta}=\frac{\partial^2 \bar{\chi}}{\partial r^2}+\frac{1}{r}\frac{\partial \bar{\chi}}{\partial r}-\frac{q^2\pi^2}{c^2}\bar{\chi}+\delta(\theta)\,\bar{s}(r,q), \tag{2.3}$$

where $\bar{\chi}\equiv\bar{\chi}(r,q,\theta)$.

Again multiplying (2.3) throughout by $rJ_0(\xi_i r)$, integrating with respect to r from 0 to a, employing the formulae [1, p. 87, (62)] and (1.3), we obtain

$$\frac{d\bar{\chi}_J}{d\theta}+\left(\frac{q^2\pi^2}{c^2}+\xi_i^2\right)\bar{\chi}_J=\delta(\theta)\,\bar{s}_J(\xi_i\,,q),$$

where $\bar{\chi}_J\equiv\bar{\chi}_J(\xi_i,q,\theta)$.

The solution of this ordinary differential equation is

$$\bar{\chi}_J=\exp\left[-\left(\frac{q^2\pi^2}{c^2}+\xi_i^2\right)\theta\right]\bar{s}_J(\xi_i,q)$$

which, with the help of the inversion formulae (1.2) and (1.4), reduces to

$$\left\{\begin{aligned}\chi(r,z,\theta)=\frac{4}{a^2c}\sum_{q=1}^{\infty}\sin\left(\frac{q\pi z}{}\right)\sum_i\frac{J_0(r\xi_i)\,\bar{s}_J(\xi_i,q)}{[J_1(a\xi_i)]^2}\,\cdot\\ \cdot\exp\left[\quad\left(\frac{q^2\pi^2}{c^2}+\xi_i^2\right)\theta\right].\end{aligned}\right. \tag{2.4}$$

Special cases: (i) If the source function $s(r,z)$ is of the form $s(r)g(z)$, where $s(r)$ and $g(z)$ are functions of r and z, respectively, then (2.4) reduces to

$$\left\{\begin{aligned}\chi(r,z,\theta)=\frac{4}{a^2c}\sum_{q=1}^{\infty}\sin\left(\frac{q\pi z}{c}\right)\bar{g}(q)\cdot\sum_i\frac{J_0(r\xi_i)\,s_J(\xi_i)}{[J_1(a\xi_i)]^2}\,\cdot\\ \cdot\exp\left[-\left(\frac{q^2\pi^2}{c^2}+\xi_i^2\right)\theta\right].\end{aligned}\right. \tag{2.5}$$

(ii) If the source of neutrons is a point source at (r',z') i.e., $s(r,z)=$ $=s_0\delta(r-r')\delta(z-z')$, then, making use of the formula [1, p. 331, (77 f)], we have

$$\bar{s}_J(\xi_i,q)=s_0\,r'\,J_0(\xi_i r')\sin\left(\frac{q\pi z'}{c}\right),$$

and (2.4) becomes

$$\begin{aligned}\chi(r,z,\theta)=\frac{4\,s_0 r'}{a^2c}\sum_{q=1}^{\infty}\sin\left(\frac{q\pi z}{c}\right)\sin\left(\frac{q\pi z'}{c}\right)\cdot\\ \cdot\sum_i\frac{J_0(\xi_i r)}{[J_1(a\xi_i)]^2}J_0(\xi_i r')\exp\left[\quad\left(\frac{q^2\pi^2}{c^2}+\xi_i^2\right)\theta\right].\end{aligned}$$

(iii) In (2.4), substituting $s(r,z)$ and $\bar{s}_J(\xi_i,q)$ from (1.6) and (1.7), we have on simplification

$$\chi(r, z, \theta) = \frac{B(\varrho, \nu)\Gamma(\frac{3}{2}-\alpha')\, 4a^{\varrho+\nu-2}}{\Gamma(\alpha')\sqrt{\pi}} \sum_{\lambda=0}^{\infty} \frac{\Gamma(\alpha'+\lambda)}{\Gamma(2-\alpha'+\lambda)} \cdot$$

$$\cdot \sin\left[(2\lambda+1)\frac{\pi z}{c}\right] \sum_i \frac{J_0(r\xi_i)}{[J_1(a\xi_i)]^2} \sum_{s=0}^{\infty} \frac{(\varrho)_{2s}\left[-\dfrac{\xi_i^2 a^2}{4}\right]^s}{(s!)^2(\varrho+\nu)_{2s}} \cdot$$

$$\cdot \exp\left[-\left\{\frac{(2\lambda+1)^2\pi^2}{c^2}+\xi_i^2\right\}\theta\right] \cdot F_{3,1}^{4,1}\left[\begin{array}{c|l} a & \alpha,\ \beta,\ \frac{3}{2}-\alpha',\ 1-\alpha' : \text{———} ;\ \varrho+2s \\ a & \varrho\ +\nu+2s,\ 1-\alpha'-\lambda,\ 2-\alpha'-\lambda : - ;\ \delta \end{array}\right],$$

provided Re $(\nu) > 0$, Re $(\varrho) > 0$ and Re $(\alpha') < 1$.

Our thanks are due to Dr. S. M. Das Gupta, Director, G. S. Institute of Technology and Science, Indore, for the facilities he provided for this work.

Pratap College,
Amalner (Jalgaon)
Maharashtra (India)

REFERENCES

1. Sneddon, I. N., Fourier transforms, McGraw-Hill, New York, 1951.
2. Erdélyi, A. et al, Integral transforms, Vol. II, New York, 1954.
3. Luke, Y. L., The Special functions and their approximations, Vol. I, Academic Press, New York, 1969.